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Vector Quasi-Hemivariational Inequalities and Discontinuous Elliptic Systems

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Abstract. We develop an existence theory for hemivariational inequalities in vector-valued function spaces which involve pseudomonotone operators. The obtained abstract result is used to study quasilinear elliptic systems whose lower order coupling vector field depends discontinuously upon the solution vector. We provide conditions that allow the identification of regions of existence of solutions for such systems, so called trapping regions.

Key words: hemivariational inequality, trapping regions

1. Introduction

Let $V \subset W^{1,r}(\Omega;\mathbb{R}^N), r > 1$, be a reflexive Banach space compactly imbedded into $L^p(\Omega;\mathbb{R}^N)$, $p > 1$. Suppose that $f_{(k)}: \Omega \times \mathbb{R}^N \to \mathbb{R}, k = 1, \dots, N$, are Baire-measurable functions, and for any $k \in \{1, \dots, N\}$, for a.e. $x \in \Omega$ and for all $(\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_N) \in \mathbb{R}^{N-1}$, the functions $\mathbb{R} \ni \xi_k \mapsto f_{(k)}(x, \xi_1, \dots, \xi_k, \dots, \xi_N)$ are locally Lipschitz. Let

$$f^0_{(k)}(x,\xi;\eta_k) = \limsup_{\substack{h\to 0\\ \lambda \to 0_+}} \Big\{ f_k(x,\xi_1,\dots,\xi_k + h + \lambda\eta_k,\dots,\xi_N) - f_k(x,\xi_1,\dots,\xi_k+h,\dots,\xi_N)\Big\}/\lambda$$

denote the partial generalized directional derivative of $f_{(k)}$ at ξ in the direction η_k, and define for each $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$ Clarke's partial generalized gradient given by

$$\partial_k f_{(k)}(x,\xi) = \{\chi \in \mathbb{R}: f^0_{(k)}(x,\xi;\eta_k) \geqslant \chi\eta_k \quad \forall \eta_k \in \mathbb{R}\}, \quad k \in \{1,\dots,N\}.$$

Suppose that $A: V \to V^\star$ is a bounded, pseudomonotone operator and let $g \in V^\star$.

Our aim is to study the following existence problem:
Problem (P). Find $u = (u_1, \dots, u_N) \in V$ satisfying the *hemivariational inequality*:

$$\langle Au - g, v - u\rangle_V + \int_\Omega \sum_{k=1}^{N} f^0_{(k)}(x, u(x); v_k(x) - u_k(x))\, dx \geqslant 0,$$
$$\forall v = (v_1, \dots, v_N) \in V.$$

Hemivariational inequalities introduced by P.D. Panagiotopoulos have attracted increasing attention over the last decade mainly due to its many applications in mechanics and engineering, cf. e.g. [23, 25]. This new type of variational inequalities arise, e.g., in mechanical problems when nonconvex, nonsmooth energy functionals (so-called superpotentials) occur, which result from nonmonotone, multivalued constitutive laws, such as for example unilateral contact and friction problems, cf. e.g. [23–25]. The theory of hemivariational inequalities extends the standard theory of variational inequalities by replacing the subdifferential of convex functionals with the directional differentiation in the sense of Clarke of nonconvex functions.

The use of topological methods for the study of hemivariational inequalities and their applications has been shown in [13–18, 20, 23–25], and the references quoted there.

Coercive and semicoercive hemivariational inequalities in vector-valued function spaces have been considered in [21, 22] under the unilateral growth condition [18].

The main goal of this paper is to develop an existence theory of quasihemivariational inequalities (cf. [23]) in vector-valued function spaces involving pseudomonotone operators, i.e., for problem (P). The obtained abstract results will then be used to study quasilinear elliptic systems whose lower order coupling vector field may depend discontinuously upon the solution vector. We provide conditions that allow the identification of regions of existence of solutions for such systems, so called trapping regions.

2. Hypotheses and Premilinary Results

Throughout this paper we shall assume the following hypotheses:

(H1) $A : V \to V^\star$ is a bounded, pseudomonotone operator, i.e. A maps bounded sets into bounded sets and that the following conditions are satisfied [2, 3]:

(i) The effective domain of A coincides with the whole V, i.e. $\mathrm{Dom}(A) = V$;

(ii) For any $\{u_n\} \subset V$, if $u_n \to u$ weakly in V and $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_V \leqslant 0$ then $\liminf_{n\to\infty} \langle Au_n, u_n - v\rangle_V \geqslant \langle Au, u - v\rangle_V$ for any $v \in V$.

(H2) There exist positive constants $a, b > 0$ and $1 \leqslant \sigma < p$ such that

$$\langle Au, u\rangle_V \geqslant a\|u\|_V^p - b\|u\|_V^\sigma, \quad \forall u \in V.$$

(H3) For any $k \in \{1, \dots, N\}$,

(i) $\mathbb{R}^N \times \mathbb{R} \ni (\xi, \eta_k) \mapsto f^0_{(k)}(x, \xi; \eta_k)$ is upper semicontinuous for a.e. $x \in \Omega$;

(ii) $\Omega \times \mathbb{R}^N \times \mathbb{R} \ni (x, \xi, \eta_k) \mapsto f^0_{(k)}(x, \xi; \eta_k)$ is Baire-measurable;

(H4) For any $R \geqslant 0$ there exists $K_R > 0$ such that the condition $|f^0_{(k)}(x, \xi; \eta_k)| \leqslant K_R|\eta_k|$, $\forall \xi \in \mathbb{R}^N$ with $|\xi| \leqslant R, \forall \eta_k \in \mathbb{R}$ and for a.e. $x \in \Omega$, is valid for $k \in \{1, \dots, N\}$.

(H5) For any $k \in \{1, \dots, N\}$ there exists a nonnegative constant $\alpha_k \geqslant 0$ with the property that

$$f^0_{(k)}(x, \xi; -\xi_k) \leqslant \alpha_k(1 + |\xi|^q), \quad \forall \xi \in \mathbb{R}^N,$$

for some $q < p$.

LEMMA 1. *Let (H4) and (H5) be satisfied. Then there exists a nondecreasing function $\alpha: \mathbb{R}_+ \to \mathbb{R}_+$ with the property that*

$$\sum_{k=1}^N f^0_{(k)}(x, \xi; \eta_k - \xi_k) \leqslant \alpha(r)(1 + |\xi|^q), \quad \forall \xi \in \mathbb{R}^N, \quad \eta \in \mathbb{R}, \quad |\eta| \leqslant r, r \geqslant 0 \quad \textit{for a.e. } x \in \Omega. \tag{1}$$

Proof. Recall that $\mathbb{R} \ni \mu \mapsto f^0_{(k)}(x, \xi; \mu)$ is positively homogeneous [10]. It is sufficient to argue for $\eta_k \neq 0$. For $0 < |\eta_k| \leqslant |\xi_k|$ the hypothesis $(H5)$ yields

$$\begin{aligned}
\sum_{k=1}^N f^0_{(k)}(x, \xi; \eta_k - \xi_k) &= \sum_{k=1}^N f^0_{(k)}\left(x, \xi; -\xi_k\left(1 - \frac{\eta_k}{\xi_k}\right)\right) \\
&= \sum_{k=1}^N \left(1 - \frac{\eta_k}{\xi_k}\right) f^0_{(k)}(x, \xi; -\xi_k) \\
&\leqslant \sum_{k=1}^N \alpha_k \left(1 - \frac{\eta_k}{\xi_k}\right)(1 + |\xi|^q) \\
&\leqslant 2\sum_{k=1}^N \alpha_k(1 + |\xi|^q),
\end{aligned}$$

while for $|\xi_k| < |\eta_k| \leqslant r, r > 0$, by $(H4)$ and $(H5)$ it follows

$$\sum_{k=1}^{N} f^0_{(k)}(x, \xi; \eta_k - \xi_k) \leqslant \sum_{k=1}^{N} |\eta_k| f^0_{(k)}\left(x, \xi; \frac{\eta_k}{|\eta_k|}\right) + \sum_{k=1}^{N} f^0_{(k)}(x, \xi; -\xi_k)$$
$$\leqslant \left(N r K_r + \sum_{k=1}^{N} \alpha_k\right)(1 + |\xi|^q).$$

The foregoing estimates imply that if we set $\alpha(r) := N r K_r + 2\sum_{k=1}^{N} \alpha_k$ then (1) is fulfilled. □

3. Finite Dimensional Approximation

Let Λ be a class of all finite dimensional subspaces of $V \cap L^\infty(\Omega; \mathbb{R}^N)$. For any $F \in \Lambda$ consider the following problem.
Problem (P_F). Find $u_F = (u_{F1}, \dots, u_{FN}) \in F$ and $\chi_F = (\chi_{F1}, \dots, \chi_{FN}) \in L^1(\Omega; \mathbb{R}^N)$ such that

$$\langle A u_F - g, v - u_F \rangle_V + \int_\Omega \chi_F \cdot (v - u_F)\, d\Omega = 0, \quad \forall v = (v_1, \dots, v_N) \in F, \tag{2}$$

$$\int_\Omega \chi_F \cdot (v - u_F)\, d\Omega \leqslant \int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u_F; v_k - u_{Fk})\, d\Omega, \quad \forall v \in L^\infty(\Omega; \mathbb{R}^N). \tag{3}$$

PROPOSITION 2. *Under the hypotheses (H1)–(H5) for any $F \in \Lambda$ the problem (P_F) has at least one solution. Moreover, a constant $M > 0$ independent of F can be found such that*

$$\|u_F\|_V \leqslant M, \quad \forall F \in \Lambda. \tag{4}$$

Proof. For $F \in \Lambda$ define $\Gamma_F : F \to 2^{L^1(\Omega; \mathbb{R}^N)}$ as

$$\Gamma_F(v) := \left\{ \chi \in L^1(\Omega; \mathbb{R}^N) : \int_\Omega \chi \cdot w\, dx \leqslant \int_\Omega \sum_{k=1}^{N} f^0_{(k)}(v; w_k)\, d\Omega, \right.$$
$$\left. \forall w \in L^\infty(\Omega; \mathbb{R}^N) \right\}. \tag{5}$$

Notice that $\Gamma_F(\cdot)$ has nonempty, convex and closed values, and if $\psi \in \Gamma_F(v)$ and $v \in F$ then

$$\|\psi\|_{L^1(\Omega;\mathbb{R}^N)} \leqslant K_{\|v\|_{L^\infty(\Omega;\mathbb{R}^N)}}.$$

Moreover, from the upper semicontinuity of $\sum_{k=1}^N f_{(k)}^0(x,\cdot;\cdot)$ and Fatou's lemma it follows that Γ_F is upper semicontinuous from F into $L^1(\Omega,\mathbb{R}^N)$ endowed with the weak topology.

Further, let $\tau_F : L^1(\Omega;\mathbb{R}^N) \to F^\star$ assigns to any $\psi \in L^1(\Omega)$ the element $\tau_F\psi \in F^\star$ defined by

$$\langle \tau_F\psi, v\rangle_F := \int_\Omega \psi \cdot v\, dx, \quad \forall v \in F. \tag{6}$$

Let us note that τ_F is a linear continuous operator from the weak topology of $L^1(\Omega;\mathbb{R}^N)$ to the (unique) linear topology on $F^\star$. Therefore, $G_F : F \to 2^{F^\star}$ given by

$$G_F(v) := \tau_F\Gamma_F(v), \quad \forall v \in F, \tag{7}$$

is upper semicontinuous.

Since F is finite dimensional, by (H1) it follows that $A_F := i_F^* A i_F$ is continuous from F into F^*. Thus, if we set $g_F := i_F^* g$ then $A_F + G_F - g_F : F \to 2^{F^\star}$ is an upper semicontinuous multivalued mapping with nonempty, bounded, closed and convex values. In addition, by (6), (7) and $(H2)$, for any $v_F \in F$ and $\psi_F \in G_F(v_F)$ one obtains the estimate

$$\begin{aligned}\langle A_F v_F + \psi_F - g_F, v_F\rangle_F &\geqslant \langle A v_F - g, v_F\rangle_V - \int_\Omega \sum_{k=1}^N f_{(k)}^0(v_F; -v_{Fk})\, d\Omega\\ &\geqslant a\|v_F\|_V^p - b\|v_F\|_V^\sigma - \|g\|_{V^\star}\|v_F\|_V\\ &\quad - \sum_{k=1}^N \alpha_k \int_\Omega (1 + |v_F|^q)\, dx\\ &\geqslant a\|v_F\|_V^p - b\|v_F\|_V^\sigma - \|g\|_{V^\star}\|v_F\|_V\\ &\quad - k|\Omega| - k\|v_F\|_V^q.\end{aligned} \tag{8}$$

Since $q, \sigma < p$, from (8) there exists a number $M > 0$ not depending on $F \in \Lambda$ such that the condition $\|v_F\|_V = M$ implies

$$\langle A_F v_F + \psi_F - g_F, v_F\rangle_F \geqslant 0. \tag{9}$$

Accordingly, inequality (9) enables us to invoke ([1], Corollary 3, p. 337) to deduce the existence of $u_F \in F$ with property (4) such that $0 \in A_F u_F +$

$G_F(u_F) - g_F$. This ensures that for some $\chi_F \in \Gamma_F(u_F)$ one has that $A_F u_F + \tau_F \chi_F - g_F = 0$, so (u_F, χ_F) is a solution of (P_F). This completes the proof. □

PROPOSITION 3. *Assume that* $(u_F, \chi_F) \in F \times L^1(\Omega; \mathbb{R}^N)$ *is a solution of* (P_F). *Then the set* $\{\chi_F\}_{F \in \Lambda}$ *is weakly precompact in* $L^1(\Omega; \mathbb{R}^N)$.

Proof. Since Ω is bounded, according to the Dunford–Pettis theorem (see, e.g., [12], p. 239) it suffices to show that for each $\varepsilon > 0$ a number $\delta > 0$ can be determined such that for any $\omega \subset \Omega$ with $|\omega| < \delta$,

$$\int_\omega |\chi_F| dx < \varepsilon, \quad \forall F \in \Lambda. \tag{10}$$

From Lemma 1 it follows that there exists a function $\alpha: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{k=1}^N f^0_{(k)}(x, \xi; \eta_k - \xi_k) \leqslant \alpha(r)(1 + |\xi|^q), \quad \forall \xi, \eta \in \mathbb{R}^N, \ |\eta| \leqslant r, \\ r \geqslant 0, \text{a.e. in } \Omega. \tag{11}$$

Fix $r > 0$ and let $\eta \in \mathbb{R}^N$ be such that $|\eta| \leqslant r$. Then, by (2) and (3), $\chi_F \cdot (\eta - u_F) \leqslant \sum_{k=1}^N f^0_{(k)}(u_F; \eta_k - u_{Fk})$, from which we get

$$\chi_F \cdot \eta \leqslant \chi_F \cdot u_F + \alpha(r)(1 + |u_F|^q) \quad \text{for a.e. } x \in \Omega. \tag{12}$$

Let us set $\eta \equiv \frac{r}{\sqrt{N}}(\mathrm{sgn}\chi_{F1}(x), \dots, \mathrm{sgn}\chi_{FN}(x))$ where $\mathrm{sgn} y = 1$ if $y > 0$, $\mathrm{sgn} y = 0$ if $y = 0$, $\mathrm{sgn} y = -1$ if $y < 0$. One obtains that $|\eta| \leqslant r$ and $\chi_F(x) \cdot \eta \geqslant \frac{r}{\sqrt{N}}|\chi_F(x)|$ for almost all $x \in \Omega$. Therefore from (12) it results

$$\frac{r}{\sqrt{N}}|\chi_F| \leqslant \chi_F \cdot u_F + \alpha(r)(1 + |u_F|^q).$$

Integrating this inequality over $\omega \subset \Omega$ yields

$$\int_\omega |\chi_F| \, d\Omega \leqslant \frac{\sqrt{N}}{r} \int_\omega \chi_F \cdot u_F \, d\Omega + \frac{\sqrt{N}}{r}\alpha(r)|\omega| \\ + \frac{\sqrt{N}}{r}\alpha(r)|\omega|^{\frac{p-q}{p}} \|u_F\|^q_{L^p(\Omega;\mathbb{R}^N)}. \tag{13}$$

Consequently, from (4) and (13) it follows that

$$\int_\omega |\chi_F| \, d\Omega \leqslant \frac{\sqrt{N}}{r} \int_\omega \chi_F \cdot u_F \, d\Omega + \frac{\sqrt{N}}{r}\alpha(r)|\omega| + \frac{\sqrt{N}}{r}\alpha(r)|\omega|^{\frac{p-q}{p}} \gamma^q M^q, \tag{14}$$

where $\gamma > 0$ is a constant satisfying $\|\cdot\|_{L^p(\Omega;\mathbb{R}^N)} \leqslant \gamma\|\cdot\|_V$.

We claim

$$\int_\omega \chi_F \cdot u_F \, d\Omega \leqslant C \tag{15}$$

for some positive constant C not depending on $\omega \subset \Omega$ and $F \in \Lambda$. Indeed, from (11) we derive that

$$\chi_F \cdot u_F + \alpha(0)(|u_F|^q + 1) \geqslant 0 \text{ for a.e. in } \Omega.$$

Thus it follows

$$\begin{aligned}\int_\omega \chi_F \cdot u_F \, d\Omega &\leqslant \int_\omega (\chi_F \cdot u_F + \alpha(0)(|u_F|^q + 1)) \, d\Omega \\ &\leqslant \int_\Omega (\chi_F \cdot u_F + \alpha(0)(|u_F|^q + 1)) \, d\Omega \\ &\leqslant \int_\Omega \chi_F \cdot u_F \, d\Omega + k_1(\|u_F\|_V^q + |\Omega|),\end{aligned}$$

where $k_1 > 0$ is a constant. By (4) and (2) (with $v = 0$) it turns out that

$$\int_\Omega \chi_F \cdot u_F \, d\Omega = -\langle Au_F - g, u_F\rangle \leqslant C,$$

The estimates above imply (15).

Further, (14) and (15) entail

$$\int_\omega |\chi_F| dx \leqslant \frac{\sqrt{N}}{r} C + \frac{\sqrt{N}}{r}\alpha(r)|\omega| + \frac{\sqrt{N}}{r}\alpha(r)|\omega|^{\frac{p-q}{p}}\gamma^q M^q, \quad \forall r > 0. \tag{16}$$

Corresponding to $\varepsilon > 0$, fix $r > 0$ with

$$\frac{\sqrt{N}}{r} C < \frac{\varepsilon}{2} \tag{17}$$

and then take $\delta > 0$ small enough to have

$$\frac{\sqrt{N}}{r}\alpha(r)|\omega| + \frac{\sqrt{N}}{r}\alpha(r)|\omega|^{\frac{p-q}{p}}\gamma^q M^q < \frac{\varepsilon}{2} \tag{18}$$

provided that $|\omega| < \delta$. Using this together with (16) and (17) it follows that (10) is justified whenever $|\omega| < \delta$. This completes the proof. □

4. Existence of Solutions

In this section we prove our main existence result which reads as follows

THEOREM 4. *Suppose that (H1)–(H5) hold. Then there exists* $(u, \chi) \in V \times L^1(\Omega; \mathbb{R}^N)$ *such that the quasi-hemivariational inequality*

$$\langle Au - g, v - u\rangle_V + \int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u; v_k - u_k)\, d\Omega \geqslant 0, \quad \forall v \in V, \tag{19}$$

is satisfied. Moreover,

$$\langle Au - g, v - u\rangle_V + \int_\Omega \chi \cdot (v - u)\, d\Omega = 0, \quad \forall v \in V \cap L^\infty(\Omega; \mathbb{R}^N), \tag{20}$$

$$\chi_k \in \partial_k f_{(k)}(u) \ \ a.e. \ in \ \Omega, \quad \chi_k u_k \in L^1(\Omega), \quad \forall k \in \{1, \dots, N\}. \tag{21}$$

Proof. The proof is divided into a sequence of steps.

Step 1. For every $F \in \Lambda$ we introduce

$$U_F = \{u_F \in F: \text{ for some } \chi_F \in L^1(\Omega; \mathbb{R}^N), (u_F, \chi_F) \text{ is a solution of } (P_F)\}$$

and

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} U_{F'}.$$

By Proposition 2, W_F is nonempty (even U_F is nonempty) and contained in the ball $B_M = \{v \in V : \|v\|_V \leqslant M\}$. We denote by weakcl$(W_F)$ the closure of W_F in the weak topology of V. The weak compactness of weakcl(W_F) follows from $W_F \subset B_M \subset V$ and the reflexivity of V. The family $\{$weakcl$(W_F)\}_{F \in \Lambda}$ has the finite intersection property. Indeed, if $F_1, \dots, F_k \in \Lambda$, then one has that $W_{F_1} \cap \dots \cap W_{F_k} \supset W_F$, with $F = F_1 + \dots + F_k$. Thus by the classical argument we conclude that there exists an element $u \in V$ with

$$u \in \bigcap_{F \in \Lambda} \text{weakcl}(W_F).$$

Let us choose $F \in \Lambda$ arbitrarily. Since V is reflexive, one can extract an increasing sequence of subspaces $\{F^{(n)}\}$, each containing F, and for each n an element $u^{(n)} \in U_{F^{(n)}}$ such that $u^{(n)} \to u$ weakly in V as $n \to \infty$ (Proposition 11, p. 274 [3]). Let us denote by $\{\chi^{(n)}\} \subset L^1(\Omega; \mathbb{R}^N)$ the corresponding sequence with the property that for each n a pair $(u^{(n)}, \chi^{(n)})$ is a solution of $(P_{F^{(n)}})$, i.e.

$$\langle Au^{(n)} - g, v - u^{(n)}\rangle_V + \int_\Omega \chi^{(n)} \cdot (v - u^{(n)})\, d\Omega = 0, \quad \forall v \in F^{(n)}, \tag{22}$$

$$\int_\Omega \chi^{(n)} \cdot (v - u^{(n)})\, d\Omega \leqslant \int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u^{(n)}; v_k - u_k^{(n)})\, d\Omega, \quad \forall v \in L^\infty(\Omega; \mathbb{R}^N). \tag{23}$$

From Proposition 3 it follows that without loss of generality we can suppose that $\chi^{(n)} \to \chi^{(F)}$ weakly in $L^1(\Omega; \mathbb{R}^N)$ for some $\chi^{(F)} \in L^1(\Omega; \mathbb{R}^N)$. Finally we have asserted that

$$u^{(n)} \to u \text{ weakly in } V \tag{24}$$

$$\chi^{(n)} \to \chi^{(F)} \text{ weakly in } L^1(\Omega; \mathbb{R}^N). \tag{25}$$

Step 2. Now we prove that $\chi^{(F)} = (\chi_k^{(F)})$ in (25) has the property that

$$\chi_k^{(F)} \in \partial_k f_{(k)}(u) \quad \text{a.e. in } \Omega, \quad k = 1, \dots, N, \tag{26}$$

which can be written equivalently as $\chi^{(F)} \in \Gamma(u)$, where

$$\Gamma(u) := \left\{ \chi \in L^1(\Omega; \mathbb{R}^N) : \int_\Omega \chi \cdot v\, dx \leqslant \int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u; v_k)\, d\Omega, \quad \forall v \in L^\infty(\Omega; \mathbb{R}^N) \right\}. \tag{27}$$

Since V is compactly imbedded into $L^p(\Omega; \mathbb{R}^N)$, due to (24) one may suppose that

$$u^{(n)} \to u \text{ strongly in } L^p(\Omega; \mathbb{R}^N). \tag{28}$$

This implies that for a subsequence of $\{u^{(n)}\}$ (again denoted by the same symbol) one gets $u^{(n)} \to u$ a.e. in Ω. Thus Egoroff's theorem can be applied from which it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with mes $\omega < \varepsilon$ can be determined such that $u^{(n)} \to u$ uniformly in $\Omega \backslash \omega$ with $u \in L^\infty(\Omega \backslash \omega; \mathbb{R}^N)$. Let $v \in L^\infty(\Omega \backslash \omega; \mathbb{R}^N)$ be an arbitrary function. From the estimate

$$\int_{\Omega \backslash \omega} \chi^{(n)} \cdot v\, d\Omega \leqslant \int_{\Omega \backslash \omega} \sum_{k=1}^{N} f^0_{(k)}(u^{(n)}; v_k)\, d\Omega$$

combined with the weak convergence in $L^1(\Omega; \mathbb{R}^N)$ of $\chi^{(n)}$ to $\chi^{(F)}$, (28) and with the upper semicontinuity of

$$L^\infty(\Omega \backslash \omega; \mathbb{R}^N) \ni u^{(n)} \longmapsto \int_{\Omega \backslash \omega} \sum_{k=1}^N f^0_{(k)}(u^{(n)}; v_k) d\Omega$$

it follows

$$\int_{\Omega \backslash \omega} \chi^{(F)} \cdot v \, d\Omega \leqslant \int_{\Omega \backslash \omega} \sum_{k=1}^N f^0_{(k)}(u; v_k) \, d\Omega, \quad \forall v \in L^\infty(\Omega \backslash \omega; \mathbb{R}^N).$$

But the last inequality amounts to saying that $\chi_k^{(F)} \in \partial_k f_{(k)}(u)$ a.e. in $\Omega \backslash \omega, k \in \{1, \ldots, N\}$. Since $|\omega| < \varepsilon$ and ε was chosen arbitrarily,

$$\chi_k^{(F)} \in \partial_k f_{(k)}(u) \text{ a.e. in } \Omega, \quad k \in \{1, \ldots, N\}. \tag{29}$$

as claimed.

Step 3. Now, it will be shown that

$$\limsup_{n \to \infty} \int_\Omega \sum_{k=1}^N f^0_{(k)}(u^{(n)}; v_k - u_k^{(n)}) \, d\Omega \leqslant \int_\Omega \sum_{k=1}^N f^0_{(k)}(u; v_k - u_k) \, d\Omega \tag{30}$$

holds for any $v \in V \cap L^\infty(\Omega; \mathbb{R}^N)$. It can be supposed that $u^{(n)} \to u$ a.e. in Ω, since $u^{(n)} \to u$ in $L^p(\Omega; \mathbb{R}^N)$. Fix $v \in L^\infty(\Omega; \mathbb{R}^N)$ arbitrarily. In view of $\chi_k^{(n)} \in \partial_k f_{(k)}(u^{(n)}), k \in \{1, \ldots, N\}$, Lemma 1 implies

$$\sum_{k=1}^N f^0_{(k)}(u^{(n)}; v_k - u_k^{(u)}) \leqslant \alpha(\|v\|_{L^\infty(\Omega; \mathbb{R}^N)})(1 + |u^{(n)}|^q). \tag{31}$$

From Egoroff's theorem it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with mes $\omega < \varepsilon$ can be determined such that $u^{(n)} \to u$ uniformly in $\Omega \backslash \omega$. One can also suppose that ω is small enough to fulfill $\int_\omega \alpha(\|v\|_{L^\infty(\Omega; \mathbb{R}^N)})(1 + |u^{(n)}|^q) \, d\Omega \leqslant \varepsilon, n = 1, 2, \ldots,$ and $\int_\omega \alpha(\|v\|_{L^\infty(\Omega; \mathbb{R}^N)})(1 + |u|^q) \, d\Omega \leqslant \varepsilon$. Hence

$$\int_\Omega \sum_{k=1}^N f^0_{(k)}(u^{(n)}; v_k - u_k^{(n)}) \, d\Omega \leqslant \int_{\Omega \backslash \omega} \sum_{k=1}^N f^0_{(k)}(u^{(n)}; v_k - u_k^{(n)}) \, d\Omega + \varepsilon$$

which by Fatou's lemma and upper semicontinuity of $f^0_{(k)}(\cdot;\cdot)$ yields

$$\limsup_{n\to\infty}\int_\Omega\sum_{k=1}^N f^0_{(k)}(u^{(n)};v_k-u^{(n)}_k)\,d\Omega\leqslant\int_{\Omega\setminus\omega}\sum_{k=1}^N f^0_{(k)}(u;v_k-u_k)\,d\Omega+\varepsilon$$
$$\leqslant\int_\Omega\sum_{k=1}^N f^0_{(k)}(u;v_k-u_k)\,d\Omega+2\varepsilon.$$

By arbitrariness of $\varepsilon>0$ one obtains (30), as required.

Step 4. Now we show that

$$\chi^{(F)}_k u_k\in L^1(\Omega),\quad k\in\{1,\dots,N\} \tag{32}$$

$$\liminf_{n\to\infty}\int_\Omega\chi^{(n)}_k u^{(n)}_k\,d\Omega\geqslant\int_\Omega\chi^{(F)}_k u_k d\Omega,\quad k\in\{1,\dots,N\}. \tag{33}$$

There exists a sequence $\{\varepsilon^{(m)}\}_{m=1}^\infty\subset L^\infty(\Omega)$ with $0\leqslant\varepsilon^{(m)}(x)\leqslant 1$ for a.e. $x\in\Omega$, such that (Lemma 2.4, p. 122, [19]):

$$\begin{aligned}&\widehat{u}^{(m)}:=(1-\varepsilon^{(m)})u\in V\cap L^\infty(\Omega;\mathbb{R}^N),\quad m=1,2,\dots,\\&\widehat{u}^{(m)}\to u\ \text{strongly in}\ V\ \text{as}\ m\to\infty.\end{aligned} \tag{34}$$

Without loss of generality it can be assumed that $\widehat{u}^{(m)}\to u$ a.e. in Ω. Since $\chi^{(F)}_k\in\partial_k f_{(k)}(u)$, one can apply (H5) to obtain $-\chi^{(F)}_k u_k\leqslant f^0_{(k)}(u;-u_k)\leqslant\alpha_k(1+|u|^q)$. Hence

$$\chi^{(F)}_k\widehat{u}^{(m)}_k=(1-\varepsilon^{(m)})\chi^{(F)}_k u_k\geqslant-\alpha_k(1+|u|^q). \tag{35}$$

This implies that the sequence $\{\chi^{(F)}_k\widehat{u}^{(m)}_k\}$ is bounded from below by integrable function and $\chi^{(F)}_k\widehat{u}^{(m)}_k\to\chi^{(F)}_k u_k$ a.e. in Ω as $m\to\infty$. On the other hand, one gets

$$\int_\Omega\chi^{(n)}_k(\widehat{u}^{(m)}_k-u^{(n)}_k)d\Omega\leqslant\int_\Omega f^0_{(k)}(u^{(n)};\widehat{u}^{(m)}_k-u^{(n)}_k)d\Omega.$$

Thus

$$\int_\Omega\chi^{(F)}_k\widehat{u}^{(m)}_k d\Omega-\liminf_{n\to\infty}\int_\Omega\chi^{(n)}_k u^{(n)}_k d\Omega$$
$$\leqslant\limsup_{n\to\infty}\int_\Omega f^0_{(k)}(u^{(n)};\widehat{u}^{(m)}_k-u^{(n)}_k)\,d\Omega,$$

and due to (30) we are led to the estimate

$$\begin{aligned}\int_\Omega \chi_k^{(F)}\widehat{u}_k^{(m)}d\Omega &\leqslant \liminf_{n\to\infty}\int_\Omega \chi_k^{(n)}u_k^{(n)}d\Omega+\int_\Omega f^0_{(k)}(u;\widehat{u}_k^{(m)}-u_k)\,d\Omega\\ &=\liminf_{n\to\infty}\int_\Omega \chi_k^{(n)}u_k^{(n)}d\Omega+\int_\Omega f^0_{(k)}(u;-\varepsilon^{(m)}u_k)\,d\Omega\\ &\leqslant \liminf_{n\to\infty}\int_\Omega \chi_k^{(n)}u_k^{(n)}d\Omega\\ &\quad+\alpha_k(0)\int_\Omega \varepsilon^{(m)}(1+|u|^q)\,d\Omega\leqslant C,\quad C=\text{const.}\end{aligned}$$

Thus by Fatou's lemma we are allowed to conclude that $\chi_k^{(F)}u_k\in L^1(\Omega), k\in\{1,\dots,N\}$, i.e. (32) holds. Taking into account that $\varepsilon^{(m)}\to 0$ a.e. in Ω as $m\to\infty$ (passing to a subsequence if necessary) we establish (33), as required.

Step 5. It will be shown that

$$\begin{cases}(Q^{(F)})\langle Au-g,v-u\rangle_V+\displaystyle\int_\Omega \chi^{(F)}\cdot(v-u)\,d\Omega=0, & \forall v\in\displaystyle\bigcup_{n=1}^\infty F^{(n)}\supset F;\\ \chi^{(F)}\in\Gamma(u).\end{cases}$$

Since A is bounded and $\{u_F\}_{F\in\Lambda}\subset\{v\in V:\|v\|_V\leqslant M\}$, there exists $K>0$ such that $\{Au_F\}_{F\in\Lambda}\subset\{l\in V^\star:\|l\|_{V^\star}\leqslant K\}$. Therefore (22) and (26) imply that

$$\left|\int_\Omega \chi^{(F)}\cdot v\,d\Omega\right|\leqslant K\|v\|_V,\quad \forall v\in\bigcup_{n=1}^\infty F^{(n)},\quad \chi^{(F)}\in\Gamma(u) \tag{36}$$

(recall that $\{F^{(n)}\}$ is an increasing sequence containing F). Further, by making use of (32) and (33) we have $\chi^{(F)}\cdot u\in L^1(\Omega)$ and

$$\begin{aligned}\limsup_{n\to\infty}\langle Au^{(n)},u^{(n)}-v\rangle_V &\leqslant \int_\Omega \chi^{(F)}\cdot(v-u)\,d\Omega\\ &\quad+\langle g,u-v\rangle_V,\quad \forall v\in\bigcup_{n=1}^\infty F^{(n)}.\end{aligned} \tag{37}$$

Since $u^{(n)}\in F^{(n)}$ and $u^{(n)}\to u$ weakly in V, the closure of $\bigcup_{n=1}^\infty F^{(n)}$ in the strong topology of V, $\overline{\bigcup_{n=1}^\infty F^{(n)}}$, must contain u. Thus there exists a sequence $\{w_i\}\subset\bigcup_{n=1}^\infty F^{(n)}$ converging strongly to u in V as $i\to\infty$. We claim that for such a sequence,

$$\int_\Omega \chi^{(F)}\cdot w_i\,d\Omega\to\int_\Omega \chi^{(F)}\cdot u\,d\Omega\quad \text{as } i\to\infty. \tag{38}$$

Indeed, let $\{\widehat{u}^{(m)}\}_{m=1}^{\infty}$ be given by (34). From (35) it follows

$$-\sum_{k=1}^{N}\alpha_k(1+|u|^q)\leqslant \chi^{(F)}\cdot\widehat{u}^{(m)}\leqslant\sum_{k=1}^{N}\left|\chi_k^{(F)}u_k\right|,\quad m=1,2,\dots, \tag{39}$$

with the bounds $-\sum_{k=1}^{N}\alpha_k(1+|u|^q)$ and $\sum_{k=1}^{N}\left|\chi_k^{(F)}u_k\right|$ being integrable in Ω. Thus there exists a constant $C>0$ such that

$$\left|\int_{\Omega}\chi^{(F)}\cdot\widehat{u}^{(m)}d\Omega\right|\leqslant C\|\widehat{u}^{(m)}\|_V,\quad m=1,2,\dots. \tag{40}$$

Denote by $\mathcal{A}$ a linear subspace spaned by $\{\widehat{u}^{(m)}\}_{m=1}^{\infty}$ and define a linear functional $\widehat{l}_{\chi^{(F)}}:\bigcup_{n=1}^{\infty}F^{(n)}+\mathcal{A}\to\mathbb{R}$ by the formula

$$\widehat{l}_{\chi^{(F)}}(v):=\int_{\Omega}\chi^{(F)}\cdot v\,d\Omega,\quad v\in\bigcup_{n=1}^{\infty}F^{(n)}+\mathcal{A}.$$

Taking into account (36) and (40), from the Hahn–Banach theorem it follows that $\widehat{l}_{\chi^{(F)}}$ admits its linear continuous extension onto $V, l_{\chi^{(F)}}\in V^{\star}$. By the dominated convergence,

$$\int_{\Omega}\chi^{(F)}\cdot\widehat{u}^{(m)}d\Omega\to\int_{\Omega}\chi^{(F)}\cdot u\,d\Omega,\quad\text{as } m\to\infty,$$

so we get $l_{\chi^{(F)}}(u)=\int_{\Omega}\chi^{(F)}\cdot u d\Omega$ which, in particular, implies (38), as claimed.

Hence by making use of (37) we easily obtain

$$\limsup_{n\to\infty}\langle Au^{(n)},u^{(n)}-u\rangle_V\leqslant 0, \tag{41}$$

which in view of the pseudomonotonicity of A yields

$$\begin{aligned}&Au^{(n)}\to Au\ \text{weakly in } V^*\\&\langle Au^{(n)},u^{(n)}\rangle_V\to\langle Au,u\rangle_V.\end{aligned} \tag{42}$$

By (22) this concludes

$$\langle Au-g,v\rangle_V+\int_{\Omega}\chi^{(F)}\cdot v\,d\Omega=0,\quad\forall v\in\bigcup_{n=1}^{\infty}F^{(n)}. \tag{43}$$

Due to (38) we easily arrive at $(Q^{(F)})$, as desired.

Step 6. It remains to show that there exists $\chi \in \Gamma(u)$ with the associated linear functional defined by

$$\widehat{l}_\chi(v) := \int_\Omega \chi \cdot v d\Omega, \quad \forall v \in V \cap L^\infty(\Omega; \mathbb{R}^N),$$

admitting a continuous extension $l_\chi \in V^\star$ which fulfills the conditions:

$$Au - g + l_\chi = 0, \quad \langle l_\chi, u\rangle_V = \int_\Omega \chi \cdot u \, d\Omega. \tag{44}$$

For every $F \in \Lambda$ let us introduce

$$V^{(F)} = \{\chi^{(F)} \in L^1(\Omega; \mathbb{R}^N) : (Q^{(F)}) \text{ holds}\}$$

and

$$Z^{(F)} = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} V^{(F')}.$$

As in the proof of Proposition 3 we show that the family $\{\chi^{(F)}\}_{F\in\Lambda}$ is weakly precompact in $L^1(\Omega; \mathbb{R}^N)$. Denoting by weakcl $(Z^{(F)})$ the closure of $Z^{(F)}$ in the weak topology of $L^1(\Omega; \mathbb{R}^N)$ we prove analogously that the family $\{\text{weakcl } (Z^{(F)})\}_{F\in\Lambda}$ has the finite intersection property. Thus there exists an element $\chi \in \Gamma(u)$ such that for any $F \in \Lambda$ it holds

$$\langle Au - g, v\rangle_V + \int_\Omega \chi \cdot v d\Omega = 0, \quad \forall v \in F,$$

which leads immediately to (44), as desired.

Step 7. Now it will be shown that (19) holds for any $v \in V$. If $v \in V \cap L^\infty(\Omega; \mathbb{R}^N)$ then the assertion follows immediately from (20). Choose any $v \in V$ with $\sum_{k=1}^N f^0_{(k)}(u; v_k - u_k) \in L^1(\Omega)$. There exists a sequence $\widehat{v}^{(m)} = (1 - \varepsilon^{(m)})v$ such that $\{\widehat{v}^{(m)}\} \subset V \cap L^\infty(\Omega; \mathbb{R}^N)$, $\widehat{v}^{(m)} \to v$ strongly in V (cf. [19]). Since as already has been established,

$$\langle Au - g, \widehat{v}^{(m)} - u\rangle_V + \int_\Omega \sum_{k=1}^N f^0_{(k)}(u; \widehat{v}^{(m)}_k - u_k) \, d\Omega \geqslant 0,$$

so in order to show (19) it remains to deduce that

$$\limsup_{m\to\infty} \int_\Omega \sum_{k=1}^N f^0_{(k)}(u; \widehat{v}^{(m)}_k - u_k) d\Omega \leqslant \int_\Omega \sum_{k=1}^N f^0_{(k)}(u; v_k - u_k) d\Omega.$$

For this purpose let us observe that $\widehat{v}^{(m)} - u = (1-\varepsilon^{(m)})(v-u) + \varepsilon^{(m)}(-u)$ which combined with the convexity of $\sum_{k=1}^{N} f^0_{(k)}(u;\cdot)$ yields the estimate

$$\sum_{k=1}^{N} f^0_{(k)}(u;\widehat{v}^{(m)}_k - u_k) \leqslant (1-\varepsilon^{(m)}) \sum_{k=1}^{N} f^0_{(k)}(u; v_k - u_k) + \varepsilon^{(m)} \sum_{k=1}^{N} f^0_{(k)}(u;-u_k)$$
$$\leqslant \left| \sum_{k=1}^{N} f^0_{(k)}(u; v-u) \right| + \alpha(0)(1+|u|^q).$$

Thus Fatou's lemma implies the assertion.

Finally, it remains to consider the case in which $\sum_{k=1}^{N} f^0_{(k)}(u; v_k - u_k) \notin L^1(\Omega)$. Recall that if $\sum_{k=1}^{N} f^0_{(k)}(u; v_k - u_k) \notin L^1(\Omega)$ then according to the convention that $+\infty - \infty = +\infty$ we have

$$\int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u; v_k - u_k)\, d\Omega$$
$$= \begin{cases} +\infty & \text{if } \int_\Omega [\sum_{k=1}^{N} f^0_{(k)}(u; v_k-u_k)]^+ d\Omega = +\infty \\ -\infty & \text{if } \int_\Omega [\sum_{k=1}^{N} f^0_{(k)}(u; v_k-u_k)]^+ d\Omega < +\infty \text{ and} \\ & \quad \int_\Omega [\sum_{k=1}^{N} f^0_{(k)}(u; v_k-u_k)]^- d\Omega = +\infty, \end{cases}$$

where $r^+ := \max\{r,0\}$ and $r^- := \max\{-r,0\}$ for any $r \in \mathbb{R}$.
If $\int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u; v_k - u_k) d\Omega = +\infty$ then there is nothing to prove. If $\int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u; v_k - u_k) d\Omega = -\infty$ then we are led to the contradiction. Indeed, there exists a sequence $\widehat{v}^{(m)} = (1-\varepsilon^{(m)})v$ such that $\{\widehat{v}^{(m)}\} \subset V \cap L^\infty(\Omega;\mathbb{R}^N)$, $\widehat{v}^{(m)} \to v$ strongly in V. Since, as already has been established,

$$\langle Au - g, \widehat{v}^{(m)} - u\rangle_V + \int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u; \widehat{v}^{(m)}_k - u_k)\, d\Omega \geqslant 0,$$

we get

$$\int_\Omega \sum_{k=1}^{N} f^0_{(k)}(u; \widehat{v}^{(m)}_k - u_k) d\Omega \geqslant \langle Au-g, -\widehat{v}^{(m)} + u\rangle_V \geqslant -C, \quad C = \text{ const},$$

and consequently

$$\int_\Omega \left[\sum_{k=1}^{N} f^0_{(k)}(u; \widehat{v}^{(m)}_k - u_k) \right]^+ d\Omega \geqslant \int_\Omega \left[\sum_{k=1}^{N} f^0_{(k)}(u; \widehat{v}^{(m)}_k - u_k) \right]^- d\Omega - C. \tag{45}$$

By the hypothesis, $\int_\Omega[\sum_{k=1}^N f^0_{(k)}(u; v_k - u_k)]^- d\Omega = +\infty$ and $\int_\Omega[\sum_{k=1}^N f^0_{(k)}\text{x}$ $(u; v_k - u_k)]^+ d\Omega < +\infty$. Since

$$\sum_{k=1}^N f^0_{(k)}(u; \widehat{v}_k^{(m)} - u_k) \leqslant (1-\varepsilon^{(m)}) \sum_{k=1}^N f^0_{(k)}(u; v_k - u_k) + \varepsilon^{(m)} \sum_{k=1}^N f^0_{(k)}(u; -u_k)$$
$$\leqslant (1-\varepsilon^{(m)}) \sum_{k=1}^N f^0_{(k)}(u; v_k - u_k) + \alpha(0)(1+|u|^q),$$

so we obtain

$$\int_\Omega \left[\sum_{k=1}^N f^0_{(k)}(u; v_k - u_k)\right]^+ d\Omega \leqslant \int_\Omega \left[\sum_{k=1}^N f^0_{(k)}(u; v_k - u_k)\right]^+ d\Omega$$
$$+ \int_\Omega \alpha(0)(1+|u|^q) d\Omega \leqslant D,$$

where $D = \text{const} > 0$, which combined with (45) yields

$$\int_\Omega \left[\sum_{k=1}^N f^0_{(k)}(u; \widehat{v}_k^{(m)} - u_k)\right]^- d\Omega \leqslant C + D.$$

Thus the application of Fatou's lemma concludes

$$\int_\Omega \left[\sum_{k=1}^N f^0_{(k)}(u; v_k - u_k)\right]^- d\Omega \leqslant C + D.$$

contrary to the assumption that $\int_\Omega \sum_{k=1}^N f^0_{(k)}(u; v_k - u_k)\, d\Omega = -\infty$. This contradiction completes the proof of Theorem 4. □

5. Discontinuous Elliptic Systems and Trapping Regions

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. The theory developed in the preceding sections will be used to establish existence and comparison results of the following discontinuous elliptic system: Let $k \in \{1, \dots, N\}$ and $u = (u_1, \dots, u_N)$

$$A_k u_k + \partial_k f_{(k)}(u) \ni h_k \text{ in } \Omega, \quad u_k = 0 \text{ on } \partial\Omega, \tag{46}$$

where A_k is a second order quasilinear differential operator in divergence form of Leray–Lions type given by

$$A_k v(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i^{(k)}(x, \nabla v(x)), \quad \text{with } \nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N}\right),$$

$\partial_k f_{(k)}(u)$ is Clarke's generalized gradient of $f_{(k)}: \mathbb{R}^N \to \mathbb{R}$ with respect to the variable u_k, and $h_k \in V_0^*$. The variable s_k of $f_{(k)}(s_1, \dots, s_N)$ is called its *principal argument* and the variables s_j with $j \neq k$ are the nonprincipal arguments of $f_{(k)}$. If $s \in \mathbb{R}^N$ then we denote

$$[s]_k := (s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_N) \in \mathbb{R}^{N-1},$$
$$(\tau, [s]_k) := (s_1, \dots, s_{k-1}, \tau, s_{k+1}, \dots, s_N) \in \mathbb{R}^N.$$

Thus we may write, e.g., $f_{(k)}(s) = f_{(k)}(s_k, [s]_k)$, that is, $[s]_k$ is the collection of all nonprincipal arguments of $f_{(k)}$.

Boundary value problem (BVP for short) (46) may be considered as the multivalued extension of an elliptic system with a vector field $g = (g_{(1)}, \dots, g_{(N)})$ whose component functions $g_{(k)}(s_1, \dots, s_N)$ may be discontinuous with respect to s_k. More precisely, if $s_k \mapsto g_{(k)}(s_k, [s]_k)$ is assumed to be locally bounded, then under some measurability condition on $g_{(k)}$ and assuming the existence of one-sided limits $g_{(k)}(s_k \pm 0, [s]_k)$, it follows that the function $f_{(k)}$ defined by

$$f_{(k)}(s) := \int_0^{s_k} g(k)(\tau, [s]_k) d\tau$$

satisfies

$$\partial_k f_{(k)}(s) = [\liminf_{t \to s_k} g_{(k)}(t, [s]_k), \limsup_{t \to s_k} g_{(k)}(t, [s]_k)],$$

cf., e.g., [9]. Roughly speaking, the last equation means that the multivalued term $\partial_k f_{(k)}$ is obtained by filling in the gaps at the discontinuous points of $s_k \mapsto g_{(k)}(s_k, [s]_k)$. Since we only will assume that $f_{(k)}(s)$ is upper semicontinuous with respect to its nonprincipal variables s_j, $j \neq k$, the theory to be developed in this section will allow us to deal with discontinuous elliptic systems, whose component functions $g_{(k)}(s)$, in addition, may depend discontinuously also on the nonprincipal arguments.

The main goal of this section is to provide conditions for the vector field f of the BVP (46) that allow the identification of regions of existence of solutions for (46), so called trapping regions, which will be defined later. While for smooth vector fields f this kind of trapping principle is well known (see, e.g., [4]), things become much more involved in case of discontinuous vector fields. Semilinear discontinuous systems have been treated, e.g., in [6, 11] under very restrictive monotonicity assumptions of the governing vector field. The main result of this section is a strong extention of these papers.

5.1. NOTATIONS AND ASSUMPTIONS

We impose the following hypotheses of Leray–Lions type on the coefficient functions $a_i^{(k)}$, $i=1,\dots,N$, of the operators A_k.

(A1) Each $a_i^{(k)}: \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies Carathéodory conditions, i.e., $a_i^{(k)}(x,\xi)$ is measurable in $x\in\Omega$ for all $\xi\in\mathbb{R}^N$ and continuous in ξ for almost all $x\in\Omega$. There exist constants $c_0^{(k)}>0$ and functions $\kappa_0^{(k)}\in L^q(\Omega)$, $1/p+1/q=1$, such that

$$|a_i^{(k)}(x,\xi)| \leqslant \kappa_0^{(k)}(x)+c_0^{(k)}|\xi|^{p-1},$$

for a.e. $x\in\Omega$ and for all $\xi\in\mathbb{R}^N$.

(A2) $\sum_{i=1}^N (a_i^{(k)}(x,\xi)-a_i(x,\xi'))(\xi_i-\xi_i')>0$
for a.e. $x\in\Omega$, and for all $\xi,\xi'\in\mathbb{R}^N$ with $\xi\neq\xi'$.

(A3) $\sum_{i=1}^N a_i^{(k)}(x,\xi)\xi_i \geqslant \nu_k|\xi|^p$
for a.e. $x\in\Omega$, and for all $\xi\in\mathbb{R}^N$ with some constants $\nu_k>0$.

As a consequence of (A1) and (A2) the operators $A_k: V\to V^*$ defined by

$$\langle A_k u,\varphi\rangle := a_k(u,\varphi)=\sum_{i=1}^N \int_\Omega a_i^{(k)}(x,\nabla u)\frac{\partial\varphi}{\partial x_i}\,d\Omega,$$

are continuous, bounded, and monotone, and hence, in particular, pseudomonotone. A partial ordering in $L^p(\Omega)$ is defined by $u\leqslant w$ if and only if $w-u$ belongs to the positive cone $L_+^p(\Omega)$ of all nonnegative elements of $L^p(\Omega)$. This induces a corresponding partial ordering also in the subspace V of $L^p(\Omega)$, and if $u,\ w\in V$ with $u\leqslant w$ then

$$[u,w]=\{z\in V\,|\,u\leqslant z\leqslant w\}$$

denotes the order interval formed by u and w. If $u,\ w\in L^p(\Omega;\mathbb{R}^N)$ then a partial ordering is given by $u\leqslant w$ if and only if $u_k\leqslant w_k$, for $k\in\{1,\dots,N\}$, i.e., $L^p(\Omega;\mathbb{R}^N)$ is equipped with the componentwise partial ordering, which induces a corresponding partial ordering in the spaces $X:=W^{1,p}(\Omega;\mathbb{R}^N)$ and $X_0:=W_0^{1,p}(\Omega;\mathbb{R}^N)$.

We define the notion of weak solution of the BVP (46) as follows.

DEFINITION 1. The function $u = (u_1, \dots, u_N) \in X_0$ is called a *solution* of the BVP (46) if there is a function $\chi \in L^q(\Omega; \mathbb{R}^N)$ such that for $k \in \{1, \dots, N\}$

(i) $\chi_k(x) \in \partial_k f_{(k)}(u(x))$ for a.e. $x \in \Omega$,
(ii) $\langle A_k u_k, \varphi\rangle + \int_\Omega \chi_k(x)\varphi(x) d\Omega = \langle h_k, \varphi\rangle, \ \forall \varphi \in V_0$.

Let $\mathcal{R} = [\underline{u}, \overline{u}] \subset X$ be the rectangle formed by the ordered pair $\underline{u}$, $\overline{u} \in X$. The following definition is a natural extension to systems of multivalued elliptic equations of the well known notion of super- and subsolution for scalar equations. Denote $Au := (A_1 u_1, \dots, A_N u_N)$ and $\partial f(u) := (\partial_1 f_1(u), \dots, \partial_N f_N(u))$, then the BVP (46) may be rewritten in the form:

$$u \in X_0: \quad Au + \partial f(u) \ni h \text{ in } X_0^*, \tag{47}$$

where $h = (h_1, \dots, h_N) \in X_0^*$.

DEFINITION 2. The vector $Au + \partial f(u)$ is called a *generalized outward pointing vector* on the boundary $\partial\mathcal{R}$ of the rectangle $\mathcal{R}$ if for all $s \in \mathcal{R}$ the following inequalities are satisfied:

$$\langle A_k \overline{u}_k + \bar{\chi}_k, \varphi\rangle \geqslant \langle h_k, \varphi\rangle, \quad \forall \varphi \in V_0 \cap L_+^p(\Omega),$$

where $\bar{\chi}_k \in L^q(\Omega)$ and $\bar{\chi}_k(x) \in \partial_k f_{(k)}(\bar{u}_k(x), [s(x)]_k)$, and

$$\langle A_k \underline{u}_k + \underline{\chi}_k, \varphi\rangle \leqslant \langle h_k, \varphi\rangle, \quad \forall \varphi \in V_0 \cap L_+^p(\Omega),$$

where $\underline{\chi}_k \in L^q(\Omega)$ and $\underline{\chi}_k(x) \in \partial_k f_{(k)}(\underline{u}_k(x), [s(x)]_k)$.

Finally, we introduce the notion of the trapping region.

DEFINITION 3. The rectangle $\mathcal{R}$, is called a *trapping region* of the BVP (46) if $Au + \partial f(u)$ is a generalized outward pointing vector on the boundary $\partial\mathcal{R}$ and if the vector functions $\underline{u}$ and $\overline{u}$ satisfy $\underline{u} \leqslant 0 \leqslant \overline{u}$ on $\partial\Omega$.

EXAMPLE. Consider the following 2×2 system

$$-\Delta u_1 + \partial_1(-u_1)^+ \ni g_1(u_2) \text{ in } \Omega, \quad u_1 = 0 \text{ on } \partial\Omega,$$
$$-\Delta u_2 + \partial_2(-u_2)^+ \ni g_2(u_1) \text{ in } \Omega, \quad u_2 = 0 \text{ on } \partial\Omega,$$

where $w^+ := \max\{w, 0\}$, and $g_i : \mathbb{R} \to \mathbb{R}$ are assumed to be bounded, i.e., $|g_i(s)| \leqslant d_i$, $i = 1, 2$, for all $s \in \mathbb{R}$ with some nonnegative constants d_i. The corresponding nonlinearities $f_{(i)}$ in this case are given by

$$f_{(1)}(u_1, u_2) = (-u_1)^+ - u_1 g_1(u_2) + c_1,$$
$$f_{(2)}(u_1, u_2) = (-u_2)^+ - u_2 g_2(u_1) + c_2,$$

where c_i are some additive constants. Let us define $\overline{u}=(\overline{u}_1,\overline{u}_2)$ as the vector function whose components $\overline{u}_i$ are given by the unique nonnegative solution of the following Dirichlet problem $(i=1,2)$:

$$-\Delta\overline{u}_i=d_i+1 \text{ in } \Omega, \quad \overline{u}_i=0 \text{ on } \partial\Omega.$$

Similarly we define $\underline{u}=(\underline{u}_1,\underline{u}_2)$ as the vector function whose components $\underline{u}_i$ are given by the unique nonpositive solution of the following Dirichlet problem $(i=1,2)$:

$$-\Delta\underline{u}_i=-d_i \text{ in } \Omega, \quad \underline{u}_i=0 \text{ on } \partial\Omega.$$

Then one verifies that the rectangle $\mathcal{R}=[\underline{u},\overline{u}]$ is a trapping region of this system.

Let $\mathcal{R}\subset X$ be a trapping region of (46). We assume the following hypotheses for the vector field f of (46):

(F1) For $k\in\{1,\dots,N\}$ and for all $s\in\mathbb{R}^N$

(i) $[s]_k\mapsto f_{(k)}(s_k,[s]_k)$ is upper semicontinuous, uniformly with respect to s_k;

(ii) $s_k\mapsto f_{(k)}(s_k,[s]_k)$ is locally Lipschitz;

(F2) There exist constants $\alpha_k>0$, and $c_k>0$ such that

$$\eta_k^{(1)}\leqslant\eta_k^{(2)}+c_k\left(s_k^{(2)}-s_k^{(1)}\right)^{p-1} \tag{48}$$

with $\eta_k^{(i)}\in\partial_k f_{(k)}\left(s_k^{(i)},[s]_k\right)$, $i=1,2$, and for $s_k^{(1)}$, $s_k^{(2)}$ satisfying $\underline{u}_k(x)-\alpha_k\leqslant s_k^{(1)}<s_k^{(2)}\leqslant\bar{u}_k(x)+\alpha_k$, and for all $[s]_k$ satisfying $[\underline{u}(x)]_k\leqslant[s]_k\leqslant[\bar{u}(x)]_k$.

(F3) There are functions $\varrho_k\in L_+^q(\Omega)$ such that

$$\chi_k(x)\in\partial_k f_{(k)}(s_k(x),[s(x)]_k):\ |\chi_k(x)|\leqslant\varrho_k(x), \quad x\in\Omega \tag{49}$$

for all $s_k\in[\underline{u}_k-\alpha_k,\bar{u}_k+\alpha_k]$ and $[s]_k\in[[\underline{u}]_k,[\bar{u}]_k]$, where α_k is as in (F2).

Remark 5. As will be seen from the proof of our main result the upper semicontinuity imposed by (F1) is actually only required to hold with respect to the region

$$[\underline{u}_k-\alpha_k,\bar{u}_k+\alpha_k]\times[[\underline{u}]_k,[\bar{u}]_k]\subset L^p(\Omega;\mathbb{R}^N).$$

By hypothesis (F2) we impose locally a one-sided growth of $\partial_k f_{(k)}$ with respect to its principal argument, which is trivially satisfied if, e.g., $f_{(k)}$

is convex with respect to its principal argument. By hypothesis (F3) we assume a local L^q-boundedness of Clarke's gradient with respect to a slightly enlarged rectangle $\mathcal{R}_\alpha = [\underline{u} - \alpha, \bar{u} + \alpha]$.

The main result of this section is the following existence and enclosure result for the BVP (46).

THEOREM 6. *Let $\mathcal{R} = [\underline{u}, \bar{u}]$ be a trapping region, and let hypotheses (A1)–(A3) and (H1)–(H3) be fulfilled. Then the BVP (46) has a solution $u \in X_0$ within $\mathcal{R}$.*

Remark 7. The result of Theorem 6 may be extended to more general BVP, which include Leray–Lions operators A_k with coefficients $a_i^{(k)}(x, u, \nabla u)$, i.e., with coefficients that depend, in addition, on u. Moreover, the vector field f may be of the form $f = f(x, u)$, i.e., it may depend on the space variable x as well. Only for simplifying our presentation and in order to emphasize the main idea we have restricted to the BVP in the form (46).

In the proof of our main result the following truncation operators will be used:

$$(T_k u_k)(x) = \begin{cases} \bar{u}_k(x) & \text{if } u_k(x) > \bar{u}_k(x), \\ u_k(x) & \text{if } \underline{u}_k(x) \leqslant u_k(x) \leqslant \bar{u}_k(x), \\ \underline{u}_k(x) & \text{if } u_k(x, t) < \underline{u}_k(x), \end{cases} \tag{50}$$

and with $\alpha > 0$ given in (F2) (ii) we define the truncation operator T_k^α by

$$(T_k^\alpha u_k)(x) = \begin{cases} \bar{u}_k(x) + \alpha_k & \text{if } u_k(x) > \bar{u}_k(x) + \alpha_k, \\ u_k(x) & \text{if } \underline{u}_k(x) - \alpha_k \leqslant u_k(x) \leqslant \bar{u}_k(x) + \alpha_k, \\ \underline{u}_k(x) - \alpha_k & \text{if } u_k(x) < \underline{u}_k(x) - \alpha_k. \end{cases} \tag{51}$$

It is known that the truncation operators T_k, and T_k^α are continuous and bounded from V into V (see, e.g., [5, Chap. C.4]). The related truncated vector functions Tu and $T^\alpha u$ are given by $Tu = (T_1 u_1, \ldots, T_N u_N)$ and $T^\alpha u = (T_1^\alpha u_1, \ldots, T_N^\alpha u_N)$, respectively. Next we introduce cut-off functions $b_k : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$b_k(x, s) = \begin{cases} (s - \bar{u}_k(x))^{p-1} & \text{if } s > \bar{u}_k(x), \\ 0 & \text{if } \underline{u}_k(x) \leqslant s \leqslant \bar{u}_k(x), \\ -(\underline{u}_k(x) - s)^{p-1} & \text{if } s < \underline{u}_k(x). \end{cases} \tag{52}$$

Let $c_k > 0$ be generic constants. One readily verifies that b_k is a Carathéodory function satisfying the growth condition

$$|b_k(x,s)| \leqslant \varrho_k(x) + c_k |s|^{p-1} \tag{53}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, with some functions $\varrho_k \in L^q(\Omega)$. Moreover, one has the following estimate

$$\int_\Omega b_k(x, u_k(x)) u_k(x)\, d\Omega \geqslant c_k \|u_k\|^p_{L^p(\Omega)} - c_k, \quad \forall u_k \in L^p(\Omega). \tag{54}$$

In view of (53) the Nemytskij operator $B_k : L^p(\Omega) \to L^q(\Omega)$ defined by

$$B_k u_k(x) = b_k(x, u_k(x))$$

is continuous and bounded.

5.2. AUXILIARY TRUNCATED BVP

We consider first the following auxiliary truncated BVP: Find $u = (u_1, \ldots, u_N) \in X_0$ such that

$$A_k u_k + \lambda_k b_k(\cdot, u_k) + \partial_k f_{(k)}(T_k^\alpha u_k, [Tu]_k) \ni h_k, \text{ in } V_0^*, \tag{55}$$

where $\lambda_k > 0$ are some constants to be specified later. If we define

$$\tilde{b}_k(x,s) = \int_0^s b_k(x,\tau) d\tau,$$

then the function $\widetilde{f_{(k)}}$ defined by

$$\widetilde{f_{(k)}}(\cdot, s_k, [s]_k) := \lambda_k \tilde{b}_k(\cdot, s_k) + f_{(k)}(T_k^\alpha s_k, [Ts]_k) \tag{56}$$

satisfies

$$\partial_k \widetilde{f_{(k)}}(\cdot, s_k, [s]_k) = \lambda_k b_k(\cdot, s_k) + \partial_k f_{(k)}(T_k^\alpha s_k, [Ts]_k),$$

and problem (55) becomes

$$A_k u_k + \partial_k \widetilde{f_{(k)}}(\cdot, u_k, [u]_k) \ni h_k, \text{ in } V_0^*. \tag{57}$$

Due to the continuity and boundedness of the operators T_k^α, T_k, and B_k, and in view of hypotheses (F1), (F3) one can see that $\widetilde{f_{(k)}}$ fulfill the assumptions (H3)–(H5) of Section 1. Moreover, assumptions (A1)–(A3) imposed on A_k imply that the operator $A: X_0 \to X_0^*$ defined by

$Au = (A_1 u_1, \ldots, A_n u_N)$ satisfies also assumptions (H1) and (H2) of Section 1. Therefore we can apply Theorem 4 of Section 4 to the BVP (57) which yields the existence of $(u, \tilde{\chi}) \in X_0 \times L^1(\Omega; \mathbb{R}^N)$ such that

$$\langle Au, v-u \rangle + \int_\Omega \tilde{\chi} \cdot (v-u)\, d\Omega = \langle h, v-u \rangle, \quad \forall v \in X_0 \cap L^\infty(\Omega; \mathbb{R}^N), \quad (58)$$

where $\tilde{\chi}_k(x) \in \partial_k \widetilde{f_{(k)}}(u(x))$. for a.e. $x \in \Omega$, and $\tilde{\chi}_k u_k \in L^1(\Omega)$. By definition of $\widetilde{f_{(k)}}$ we get

$$\tilde{\chi}_k - \lambda_k b_k(\cdot, u_k) \in \partial_k f_{(k)}(T_k^\alpha u_k, [Tu]_k), \quad (59)$$

which by (F3) implies that

$$|\tilde{\chi}_k(x) - \lambda_k b_k(x, u_k(x))| \leqslant \varrho_k(x),$$

and hence it follows that $\tilde{\chi}_k \in L^q(\Omega)$, because $b_k(\cdot, u_k) \in L^q(\Omega)$. Since $\tilde{\chi} \in L^q(\Omega; \mathbb{R}^N)$, it follows that (58) is true for any $v \in X_0$, which shows that $(u, \tilde{\chi}) \in X_0 \times L^q(\Omega; \mathbb{R}^N)$ satisfies

$$\langle Au, v \rangle + \int_\Omega \tilde{\chi} \cdot v\, d\Omega = \langle h, v \rangle,$$

or equivalently

$$A_k u_k + \lambda_k b_k(\cdot, u_k) + \chi_k = h_k, \ \text{ in } V_0^*, \quad (60)$$

where $\chi_k = \tilde{\chi}_k - \lambda_k b_k(\cdot, u_k) \in L^q(\Omega)$, and $\chi_k(x) \in \partial_k f_{(k)}(T_k^\alpha u_k(x), [Tu(x)]_k)$ a.e. in Ω. Thus we have proved the following result.

LEMMA 8. *The auxiliary BVP (55) possesses a solution* $(u, \chi) \in X_0 \times L^q(\Omega; \mathbb{R}^N)$ *in the sense of Definition 1.*

We remark that in proving the existence for the auxiliary problem (55), actually an additional regularization technique similar as in [8] has to be applied to compensate the lack of a chain rule of Clarke's gradient with the truncation function. We have dropped this regularization technique here in order to avoid too much technicalities.

5.3. PROOF OF THEOREM 6

Proof. Theorem 6 is proved provided we are able to show that any solution u of the auxiliary BVP (55) is enclosed by the trapping region, i.e., $u \in \mathcal{R} = [\underline{u}, \bar{u}]$. This is because if $u \in \mathcal{R}$ then $b_k(\cdot, u_k) = 0$, and $f_{(k)}(T_k^\alpha u_k, [Tu]_k) =$

$f_{(k)}(u)$, and thus u is a solution of the original BVP (46), which belongs to $\mathcal{R}$. Now let u be a solution of the BVP (55). We are going to prove $u \leqslant \bar{u}$.

By definition of the trapping region $(\bar{u}, \bar{\chi}) \in X \times L^p(\Omega; \mathbb{R}^N)$ satisfies: $\bar{u} \geqslant 0$ on $\partial\Omega$ and for all $\varphi \in V_0 \cap L^p_+(\Omega)$

$$A_k \bar{u}_k + \bar{\chi}_k \geqslant h_k, \text{ in } V_0^*, \quad \text{with } \bar{\chi}_k \in \partial_k f_{(k)}(\bar{u}_k, [s]_k) \tag{61}$$

for any $s \in \mathcal{R}$. Thus (61) holds, in particular, for $s = Tu \in \mathcal{R}$ with u being the solution of (60). Subtracting (61) from (60) we obtain for any $\varphi \in V_0 \cap L^p_+(\Omega)$ the inequality

$$\langle A_k u_k - A_k \bar{u}_k, \varphi \rangle + \lambda_k \int_\Omega b_k(\cdot, u_k)\varphi d\Omega + \int_\Omega (\chi_k - \bar{\chi}_k)\varphi d\Omega \leqslant 0. \tag{62}$$

Taking in (62) the special test function $\varphi = \max\{u_k - \bar{u}_k, 0\} =: (u_k - \bar{u}_k)^+$ we get

$$\begin{aligned} &\sum_{i=1}^N \int_{\{u_k > \bar{u}_k\}} (a_i(\cdot, \nabla u_k) - a_i(\cdot, \nabla \bar{u}_k)) \frac{\partial(u_k - \bar{u}_k)}{\partial x_i} d\Omega \\ &\quad + \lambda_k \int_{\{u_k > \bar{u}_k\}} b_k(\cdot, u_k)(u_k - \bar{u}_k) d\Omega \\ &\quad + \int_{\{u_k > \bar{u}_k\}} (\chi_k - \bar{\chi}_k)(u_k - \bar{u}_k) d\Omega \leqslant 0 \end{aligned} \tag{63}$$

with $\bar{\chi}_k \in \partial_k f_{(k)}(\bar{u}_k, [Tu]_k)$, and $\chi_k \in \partial_k f_{(k)}(T_k^\alpha u_k, [Tu]_k)$, where $\{u_k > \bar{u}_k\} := \{x \in \Omega | u_k(x) > \bar{u}_k(x)\}$. If $u_k(x) > \bar{u}_k(x)$ then $T_k^\alpha u_k(x) > \bar{u}_k(x)$, and thus in view of (F2) it follows

$$\bar{\chi}_k(x) \leqslant \chi_k(x) + c_k \left(T_k^\alpha u_k(x) - \bar{u}_k(x)\right)^{p-1}. \tag{64}$$

Fox $x \in \{u_k > \bar{u}_k\}$ we have $u_k(x) \geqslant T_k^\alpha u_k(x)$, and so from (64) we obtain

$$\chi_k(x) - \bar{\chi}_k(x) \geqslant -c_k(u_k(x) - \bar{u}_k(x))^{p-1}. \tag{65}$$

By definition of the cut-off function b_k we get for the second term on the left-hand side of (63)

$$\lambda_k \int_{\{u_k > \bar{u}_k\}} b_k(\cdot, u_k)(u_k - \bar{u}_k) d\Omega = \lambda_k \int_{\{u_k > \bar{u}_k\}} (u_k(x) - \bar{u}_k(x))^p d\Omega. \tag{66}$$

By hypothesis (A2) the first term on the left-hand side of (63) is nonnegative. Thus in view of (65) and (66) from (63) we infer

$$(\lambda_k - c_k)\int_{\{u_k > \bar{u}_k\}} (u_k(x) - \bar{u}_k(x))^p d\Omega$$
$$= (\lambda_k - c_k)\int_{\Omega} \left((u_k(x) - \bar{u}_k(x))^+\right)^p d\Omega \leqslant 0, \tag{67}$$

which holds true for any $\lambda_k > 0$. Selecting the parameter λ_k such that $\lambda_k > c_k$, from (67) it follows that $(u_k - \bar{u}_k)^+ = 0$, which implies $u_k \leqslant \bar{u}_k$ a.e. in Ω. In a similar way one can show the inequality $\underline{u}_k \leqslant u_k$, which completes the proof of Theorem 6. □

Remark 9. Recently in [7] the existence of solutions within a trapping region of a 2×2 discontinuous quasilinear elliptic system has been proved under the hypothesis that the vector field is of mixed monotone type. One of the main tools used in the proof of [7] was a fixed point theorem for increasing (not necessarily continuous) mappings in ordered spaces. Since Theorem 6 of this section does not require any monotonicity of the vector field f, it provides a generalization of the result of [7].

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